

FINITE GROUPS ACTING ON HIGHER DIMENSIONAL NONCOMMUTATIVE TORI

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ABSTRACT. For the canonical action α of $\mathrm{SL}_2(\mathbb{Z})$ on 2-dimensional simple rotation algebras \mathcal{A}_θ , it is known that if F is a finite subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the crossed products $\mathcal{A}_\theta \rtimes_\alpha F$ are all AF algebras. In this paper we show that this is not the case for higher dimensional noncommutative tori. More precisely, we show that for each $n \geq 3$ there exist noncommutative simple $\phi(n)$ -dimensional tori \mathcal{A}_Θ which admit canonical action of \mathbb{Z}_n and for each odd $n \geq 7$ with $2\phi(n) \geq n + 5$ their crossed products $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ are not AF (with nonzero K_1 -groups). It is also shown that the only possible canonical action by a finite group on a 3-dimensional simple torus is the flip action by \mathbb{Z}_2 . Besides, we discuss the canonical actions by finite groups $\mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_{10}$, and \mathbb{Z}_{12} on the 4-dimensional torus of the form $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$.

1. INTRODUCTION

The *rotation algebra* \mathcal{A}_θ , $\theta \in \mathbb{R}$, is the universal C^* -algebra generated by two unitaries u_1, u_2 satisfying the commutation relation $u_2 u_1 = \exp(2\pi i \theta) u_1 u_2$. If u_1 and u_2 commute (that is, if $\theta \in \mathbb{Z}$), \mathcal{A}_θ is isomorphic to the commutative C^* -algebra $C(\mathbb{T}^2)$ of all continuous functions on the 2-dimensional torus \mathbb{T}^2 , and so the rotation algebras \mathcal{A}_θ are often called 2-dimensional noncommutative tori. If $\theta \in \mathbb{R} \setminus \mathbb{Q}$, \mathcal{A}_θ is called an *irrational rotational algebra* and this is the case exactly when \mathcal{A}_θ is a simple C^* -algebra.

More generally, for $d \geq 2$, a *noncommutative d -dimensional torus* (or simply a *d -torus*) \mathcal{A}_Θ associated with a skew symmetric real $d \times d$ matrix $\Theta = (\theta_{kj})$ is the universal C^* -algebra generated by d unitaries u_1, \dots, u_d that are subject to the commutation relations

$$u_j u_k = \exp(2\pi i \theta_{kj}) u_k u_j. \quad (1.1)$$

\mathcal{A}_Θ was introduced in [11] as the twisted group algebra $C^*(\mathbb{Z}^d, \omega_\Theta)$ of \mathbb{Z}^d twisted by the 2-cocycle ω_Θ given in (2.4).

In [16] Watatani considered an automorphism α_A , $A = (a_{ij}) \in \mathrm{SL}_2(\mathbb{Z})$, on an irrational rotational algebra \mathcal{A}_θ defined by

$$\alpha_A(u_1) = \exp(\pi i \theta a_{11} a_{21}) u_1^{a_{11}} u_2^{a_{21}}, \quad \alpha_A(u_2) = \exp(\pi i \theta a_{12} a_{22}) u_1^{a_{12}} u_2^{a_{22}} \quad (1.2)$$

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and then classified these automorphisms using the notion of K_1 -entropy. Brenken [1] used the automorphism to study representations of rotational algebras. In this paper, the action $A \mapsto \alpha_A : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{Aut}(A_\theta)$ and its d -dimensional version (Definition 2.5) will be called a *canonical action*.

The group $\mathrm{SL}_2(\mathbb{Z})$ is known to have only four (up to conjugacy) nontrivial finite subgroups which are isomorphic to \mathbb{Z}_2 , \mathbb{Z}_3 , \mathbb{Z}_4 , and \mathbb{Z}_6 . The crossed products $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_k$ of a simple \mathcal{A}_θ by the restriction of the canonical action α to \mathbb{Z}_k , $k = 2, 3, 4, 6$, are all known to be AF-algebras and moreover their K_0 groups are computed (see [3, Theorem 0.1]), which implies $\mathcal{A}_{\theta_1} \rtimes_\alpha \mathbb{Z}_k \cong \mathcal{A}_{\theta_2} \rtimes_\alpha \mathbb{Z}_l$ if and only if $k = l$ and $\theta_1 = \pm\theta_2 \bmod \mathbb{Z}$. Also it is known in the same paper [3] that $\mathcal{A}_\Theta \rtimes_\sigma \mathbb{Z}_2$ is an AF algebra if \mathcal{A}_Θ is a simple d -dimensional noncommutative torus and σ is the action given by the flip automorphism sending the unitary generators u_j to their adjoints u_j^* for $j = 1, \dots, d$. This seminal work [3] was actually motivated, as reviewed in the first chapter there, by several previous studies including, for example, the result [15] that for most irrational numbers θ , the crossed products $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_4$ are AF algebras, and it finally settled down the case $\mathcal{A}_\theta \rtimes_\alpha F$ for any (2-dimensional) irrational rotational algebras \mathcal{A}_θ and any finite groups $F \subset \mathrm{SL}_2(\mathbb{Z})$.

It would then be a very natural question to ask whether the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ of a simple higher dimensional noncommutative d -torus \mathcal{A}_Θ is still AF even when α is the canonical action of a finite subgroup G of $\mathrm{SL}_d(\mathbb{Z})$ (or $\mathrm{GL}_d(\mathbb{Z})$). But it was unclear, at least to the knowledge of the authors, even whether there are any known finite groups acting canonically on some higher dimensional noncommutative simple tori, except the flip action by \mathbb{Z}_2 , when the authors got interested in this question. The purpose of this paper is thus to find finite subgroups G of $\mathrm{GL}_d(\mathbb{Z})$ which act canonically on higher dimensional noncommutative simple d -tori \mathcal{A}_Θ and to figure out if there are simple crossed products $\mathcal{A}_\Theta \rtimes_\alpha G$ which are not AF.

In order to show that such a crossed product is not AF, it is enough to see that the K_1 -group of $\mathcal{A}_\Theta \rtimes_\alpha G$ is nonzero. From the general theory developed in [3], we can deduce without difficulty that the K -groups $K_*(\mathcal{A}_\Theta \rtimes_\alpha G)$ are equal to the K -groups $K_*(C^*(\mathbb{Z}^d \rtimes G))$ of the semidirect product group C^* -algebras.

For the finite subgroups G of $\mathrm{GL}_d(\mathbb{Z})$, we will use the companion matrix C_n of the n th cyclotomic polynomial; the matrix C_n is a $d \times d$ matrix, $d = \phi(n)$, and is of order n . The finite cyclic group $\mathbb{Z}_n = \langle C_n \rangle$ generated by C_n then acts on \mathbb{Z}^d by conjugation and we have the semidirect product group $\mathbb{Z}^d \rtimes \mathbb{Z}_n$. We show the following theorem with the aid of the recent results on topological K -theory of group C^* -algebras known in [5]:

Theorem 1.1. (Theorem 3.6) *Let $n \geq 7$ be an odd integer with $d := \phi(n)$. If $2d \geq n + 5$, then*

$$K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) \neq 0,$$

where α is the conjugation action of $\mathbb{Z}_n = \langle C_n \rangle$ on \mathbb{Z}^d .

For a prime number $n \geq 3$, it is known in [2, Theorem 0.3] that $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) = 0$ if and only if $n = 3$ or 5 .

The remaining thing to answer our question is then to show that the cyclic group $\mathbb{Z}_n = \langle C_n \rangle$ can really act canonically on some noncommutative simple d -dimensional tori \mathcal{A}_Θ , and we prove the following:

Theorem 1.2. (Theorem 4.2) *Let $n \geq 3$ and $d := \phi(n)$. Then there exist simple d -dimensional tori \mathcal{A}_Θ on which the group $\mathbb{Z}_n = \langle C_n \rangle$ acts canonically.*

From this result and Theorem 1.1, we can say that there exist many noncommutative simple higher dimensional tori whose crossed products by finite cyclic groups via canonical actions are not AF.

If $p \geq 3$ is prime (thus $d = p - 1$), we can say more on the skew symmetric $d \times d$ matrices Θ , and apply [2, Theorem 0.3] to figure out exactly when the simple crossed product $\mathcal{A}_\Theta \rtimes \mathbb{Z}_p$ is AF:

Theorem 1.3. (Theorem 4.4 and Corollary 4.5) *Let $p \geq 3$ be a prime. Then noncommutative d -tori \mathcal{A}_Θ associated with Θ of the form in (4.19) admit the canonical action of $\mathbb{Z}_p = \langle C_p \rangle$. Conversely, if $\mathbb{Z}_p = \langle C_p \rangle$ canonically acts on a d -torus \mathcal{A}_Θ , then Θ must have the form in (4.19). For the crossed product $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_p$ of a simple \mathcal{A}_Θ , it is an AF algebra if and only if $p = 3$ or 5 .*

Since the cases considered above do not include odd-dimensional noncommutative tori while 3-dimensional case, for example, is expected to be easily understood and maybe much similar to 2-dimensional tori than tori with dimension a lot higher, we examine noncommutative simple 3-tori separately and obtain the following:

Theorem 1.4. (Theorem 5.2) *The only canonical action by a nontrivial finite cyclic group on a simple 3-dimensional torus is the flip action by \mathbb{Z}_2 .*

Going one step further we will also examine the canonical actions by finite groups on 4-dimensional tori. We know from Theorem 1.3 with $p = 5$ that there are simple 4-tori admitting canonical actions by $\mathbb{Z}_5 = \langle C_5 \rangle$ and their crossed products are all AF. Among 4-dimensional simple tori, especially we will look at \mathcal{A}_Θ isomorphic to the tensor product $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ of a simple 2-dimensional torus \mathcal{A}_θ with itself and describe explicitly in Proposition 6.4 the canonical actions by the groups $\mathbb{Z}_n = \langle C_n \rangle$ for $n = 5, 8, 10, 12$.

This paper is organized as follows. In Section 2, we first review definitions and important results which we need in later chapters and then make it clear why the resulting crossed product under consideration in this paper is AF exactly when its K_1 group is zero. In Section 3 we explain how the machinery obtained in [3] and the results in [2, 5] can be applied to our situation, and then prove Theorem 1.1. In Section 4, we prove Theorem 1.2 and Theorem 1.3. Then applying the complete list of elements in $\text{GL}_3(\mathbb{Z})$ of finite order known in [14], we prove Theorem 1.4 in Section 5. Finally in Section 6, we examine in Proposition 6.4

some of the 4-dimensional simple tori and the canonical actions by the finite groups $\mathbb{Z}_5, \mathbb{Z}_8, \mathbb{Z}_{10}, \mathbb{Z}_{12}$.

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2. PRELIMINARIES

In this section we recall basic definitions and important facts (from [2, 3, 5, 6, 7, 10, 11, 12]) which shall be used throughout the paper.

2.1. Twisted group algebras of discrete groups. Let G be a second countable discrete group. A 2-cocycle on G is a function $\omega : G \times G \rightarrow \mathbb{T}$ such that $\omega(x, y)\omega(xy, z) = \omega(y, z)\omega(x, yz)$ and $\omega(x, 1) = \omega(1, x) = 1$ for $x, y, z \in G$. By $\ell^1(G, \omega)$ we denote the twisted convolution $*$ -algebra of all summable functions on G with

$$(f *_\omega g)(x) = \sum_{y \in G} f(y)g(y^{-1}x)\omega(y, y^{-1}x)$$

$$f^*(x) = \overline{\omega(x, x^{-1})f(x^{-1})}.$$

We call a map $v : G \rightarrow \mathcal{U}(\mathcal{H})$ of G into the unitary group of a Hilbert space \mathcal{H} an ω -representation of G if

$$v(x)v(y) = \omega(x, y)v(xy) \quad (2.3)$$

for $x, y \in G$. The *regular ω -representation* of G is the ω -representation $l_\omega : G \rightarrow \mathcal{U}(\ell^2(G))$ given by

$$(l_\omega(x)\xi)(y) = \omega(x, x^{-1}y)\xi(x^{-1}y)$$

for $\xi \in \ell^2(G)$ and $x, y \in G$. Every ω -representation $v : G \rightarrow \mathcal{U}(\mathcal{H})$ induces a contractive $*$ -homomorphism $v : \ell^1(G, \omega) \rightarrow B(\mathcal{H})$ (also denoted v) given by $v(f) = \sum_x f(x)v(x)$ for $f \in \ell^1(G, \omega)$, and every nondegenerate representation of $\ell^1(G, \omega)$ arises in this way. The *full twisted group algebra* $C^*(G, \omega)$ is then defined to be the enveloping C^* -algebra of $\ell^1(G, \omega)$ and the *reduced twisted group algebra* $C_r^*(G, \omega)$ is the image of $C^*(G, \omega)$ under the regular representation l_ω . If G is amenable, $C^*(G, \omega)$ is equal to $C_r^*(G, \omega)$ by [7, Theorem 3.11], and hence by (2.3)

$$C^*(G, \omega) = \overline{\text{span}}\{l_\omega(x) : x \in G\}.$$

2.2. Noncommutative tori. A real skew symmetric $d \times d$ matrix Θ induces a 2-cocycle $\omega_\Theta : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{T}$ given by

$$\omega_\Theta(x, y) = \exp(\pi i \langle \Theta x, y \rangle) \quad (2.4)$$

for $x, y \in \mathbb{Z}^d$. The twisted group algebra $C^*(\mathbb{Z}^d, \omega_\Theta)$ is called a *noncommutative d -torus* ([11]). If $\{e_i\}_{i=1}^d$ is the standard basis of \mathbb{Z}^d , then $\omega_\Theta(e_j, e_k) = \exp(\pi i \theta_{kj})$ and $C^*(\mathbb{Z}^d, \omega_\Theta) = C^*\{l_\Theta(e_i) : i = 1, \dots, d\}$, where $l_\Theta : \mathbb{Z}^d \rightarrow \mathcal{U}(\ell^2(\mathbb{Z}^d))$ denotes the regular ω_Θ -representation. It is also easy to see that for

$x, y \in \mathbb{Z}^d$, $\omega_\Theta(x, x) = 1$ and $l_\Theta(x)l_\Theta(y) = \omega_\Theta(x, y)l_\Theta(x+y) = \omega_\Theta(x, y)l_\Theta(y+x) = \omega_\Theta(x, y)\overline{\omega_\Theta(y, x)}l_\Theta(y)l_\Theta(x) = \omega_\Theta(x, y)^2l_\Theta(y)l_\Theta(x)$. Thus

$$l_\Theta(e_j)l_\Theta(e_k) = \omega_\Theta(e_j, e_k)^2l_\Theta(e_k)l_\Theta(e_j) = \exp(2\pi i\theta_{kj})l_\Theta(e_k)l_\Theta(e_j) \quad (2.5)$$

follows for $j, k = 1, \dots, d$, and

$$l_\Theta(y) = \exp\left(\pi i \sum_{k=2}^d \sum_{j=1}^{k-1} y_k y_j \theta_{jk}\right) l_\Theta(e_1)^{y_1} \cdots l_\Theta(e_d)^{y_d} \quad (2.6)$$

holds for $y = (y_1, \dots, y_d) \in \mathbb{Z}^d$. The relation (2.5) shows that the generating unitaries $\{l_\Theta(e_j)\}_{j=1}^d$ of $C^*(\mathbb{Z}^d, \omega_\Theta)$ satisfy the relation (1.1). In fact $C^*(\mathbb{Z}^d, \omega_\Theta)$ is characterized as the universal C^* -algebra generated by d unitaries $\{u_j\}_{j=1}^d$ satisfying the relations (1.1) ([11]). Also, $C^*(\mathbb{Z}^d, \omega_\Theta)$ is usually denoted by \mathcal{A}_Θ ;

$$\mathcal{A}_\Theta = C^*(\mathbb{Z}^d, \omega_\Theta). \quad (2.7)$$

For $\theta \in \mathbb{R}$, the rotation algebra \mathcal{A}_θ is the noncommutative 2-torus \mathcal{A}_Θ associated to the real skew symmetric 2×2 matrix $\Theta = (\theta_{kj})$ with $\theta_{12} = \theta$. Of course, \mathcal{A}_Θ is not necessarily noncommutative as the generators commute each other if $\theta_{kj} \in \mathbb{Z}$ for all $k, j = 1, \dots, d$.

Notation 2.1. As in [12], we use the following notation:

$$\mathcal{T}_d(\mathbb{R}) := \{ \Theta \in M_d(\mathbb{R}) : \Theta^t = -\Theta \},$$

where Θ^t denotes the transpose of Θ . Similarly, $\mathcal{T}_d(\mathbb{Z})$ denotes the set of all $d \times d$ skew symmetric matrices with entries from \mathbb{Z} . For a skew symmetric matrix $\Theta \in \mathcal{T}_d(\mathbb{R})$, we will consider the group

$$G_\Theta := \{ A \in \text{GL}_d(\mathbb{Z}) : \Theta = A^t \Theta A \}.$$

Actually it is the isotropy group of $\Theta \in \mathcal{T}_d(\mathbb{R})$ under the action of $\text{GL}_d(\mathbb{Z})$,

$$(A, \Theta) \mapsto (A^{-1})^t \Theta A^{-1} : \text{GL}_d(\mathbb{Z}) \times \mathcal{T}_d(\mathbb{R}) \rightarrow \mathcal{T}_d(\mathbb{R}).$$

We call $\Theta \in \mathcal{T}_d(\mathbb{R})$ *nondegenerate* if whenever $x \in \mathbb{Z}^d$ satisfies $\exp(2\pi i \langle x, \Theta y \rangle) = 1$ for all $y \in \mathbb{Z}^d$, then $x = 0$. Otherwise Θ is called *degenerate*. For the simplicity of the algebra \mathcal{A}_Θ , the following is known.

Theorem 2.2. ([8, Theorem 1.9], [13, Theorem 3.7]) *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$. Then the noncommutative d -torus \mathcal{A}_Θ is simple if and only if Θ is nondegenerate.*

2.3. Canonical action on noncommutative tori \mathcal{A}_Θ . For a matrix $A \in \text{GL}_d(\mathbb{Z})$, the unitary $U_A \in U(\ell^2(\mathbb{Z}^d))$ given by

$$(U_A \xi)(x) = \xi(A^{-1}x)$$

for $\xi \in \ell^2(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$, defines an automorphism $\text{Ad } U_A$ of $B(\ell^2(\mathbb{Z}^d))$. Any restrictions of $\text{Ad } U_A$ to subalgebras of $B(\ell^2(\mathbb{Z}^d))$ will also be written as $\text{Ad } U_A$.

Remark 2.3. Consider the d -torus $\mathcal{A}_\Theta = \overline{\text{span}}\{l_\Theta(x) : x \in \mathbb{Z}^d\} \subset B(\ell^2(\mathbb{Z}^d))$ associated with $\Theta \in \mathcal{T}_d(\mathbb{R})$.

- (1) If $A \in \text{GL}_d(\mathbb{Z})$ satisfies $\Theta = (A^{-1})^t \Theta A^{-1}$, it defines an automorphism $\text{Ad } U_A \in \text{Aut}(\mathcal{A}_\Theta)$ because $\text{Ad } U_A(l_\Theta(y)) = l_{(A^{-1})^t \Theta A^{-1}}(Ay) \in \mathcal{A}_\Theta$ for the generators $\{l_\Theta(y) : y \in \mathbb{Z}^d\}$ of \mathcal{A}_Θ . In fact, for $\xi \in \ell^2(\mathbb{Z}^d)$ and $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} \text{Ad } U_A(l_\Theta(y))(\xi)(x) &= (U_A l_\Theta(y) U_A^*)(\xi)(x) \\ &= (l_\Theta(y) U_A^*)(\xi)(A^{-1}x) \\ &= \omega_\Theta(y, -y + A^{-1}x)(U_A^*(\xi))(-y + A^{-1}x) \\ &= \omega_\Theta(y, -y + A^{-1}x) \xi(-Ay + x) \\ &= \omega_\Theta(y, A^{-1}x) \xi(-Ay + x) \\ &= \omega_{(A^{-1})^t \Theta A^{-1}}(Ay, x) \xi(-Ay + x) \\ &= (l_{(A^{-1})^t \Theta A^{-1}}(Ay))(\xi)(x). \end{aligned}$$

Thus we are concerned with the group G_Θ .

- (2) If $\Theta = (\theta_{ij}) \in \mathcal{T}_2(\mathbb{R})$ with $\theta := \theta_{12} \neq 0$, then $G_\Theta = \text{SL}_2(\mathbb{Z})$, which is immediate from the fact that $(A^{-1})^t \Theta A^{-1} = \det(A)\Theta$ for $A \in \text{GL}_2(\mathbb{Z})$.
(3) More generally, any matrix $A \in \text{GL}_d(\mathbb{Z})$ satisfying

$$K_A := \Theta - (A^{-1})^t \Theta A^{-1} \in \mathcal{T}_d(\mathbb{Z})$$

defines an automorphism $\tau_{K_A} \circ \text{Ad } U_A \in \text{Aut}(\mathcal{A}_\Theta)$ such that

$$(\tau_{K_A} \circ \text{Ad } U_A)(l_\Theta(y)) = l_\Theta(Ay)$$

for $y \in \mathbb{Z}^d$, where $\tau_{K_A}(l_{(A^{-1})^t \Theta A^{-1}}(y)) := l_{(A^{-1})^t \Theta A^{-1} + K_A}(y)$. The set $\{A \in \text{GL}_d(\mathbb{Z}) : \Theta - (A^{-1})^t \Theta A^{-1} \in \mathcal{T}_d(\mathbb{Z})\}$ forms a group.

For each $\Theta \in \mathcal{T}_d(\mathbb{R})$, the group G_Θ acts on \mathbb{Z}^d via matrix multiplication

$$(A, x) \mapsto Ax : G_\Theta \times \mathbb{Z}^d \rightarrow \mathbb{Z}^d$$

which then defines the semidirect product group $\mathbb{Z}^d \rtimes G_\Theta$ with the group multiplication

$$(x, A)(y, B) = (x + Ay, AB)$$

for $x, y \in \mathbb{Z}^d$ and $A, B \in G_\Theta$. Note that the cocycle ω_Θ , given in (2.4), is *invariant* under the above action; $\omega_\Theta(Ax, Ay) = \omega_\Theta(x, y)$ for $A \in G_\Theta$ and $x, y \in \mathbb{Z}^d$.

The following lemma is a special case of [7, Theorem 4.1] (see [3, Lemma 2.1]).

Lemma 2.4. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ and G be a subgroup of G_Θ . Then:*

- (1) *There is a 2-cocycle $\tilde{\omega}_\Theta$ of $\mathbb{Z}^d \rtimes G$ defined by*

$$\tilde{\omega}_\Theta((x, A), (y, B)) = \omega_\Theta(x, Ay). \quad (2.8)$$

- (2) *There is an action $\alpha : G \rightarrow \text{Aut}(\mathcal{A}_\Theta)$ given by $\alpha_A(f)(x) = f(A^{-1}x)$ for $f \in \ell^1(\mathbb{Z}^d, \omega_\Theta)$ and $A \in G$, or equivalently*

$$\alpha_A(l_\Theta(x)) = l_\Theta(Ax) \quad (l_\Theta(x) \in \mathcal{A}_\Theta).$$

(3) *There are isomorphisms*

$$\begin{aligned} C^*(\mathbb{Z}^d \rtimes G, \tilde{\omega}_\Theta) &\cong C^*(\mathbb{Z}^d, \omega_\Theta) \rtimes_\alpha G, \\ C_r^*(\mathbb{Z}^d \rtimes G, \tilde{\omega}_\Theta) &\cong C^*(\mathbb{Z}^d, \omega_\Theta) \rtimes_{\alpha, r} G \end{aligned}$$

given by $f \mapsto \Phi(f) : \ell^1(\mathbb{Z}^d \rtimes G, \tilde{\omega}_\Theta) \rightarrow \ell^1(G, \ell^1(\mathbb{Z}^d, \omega_\Theta))$ on the level of ℓ^1 -functions, where $\Phi(f)(A) = f(\cdot, A)$ for $A \in G$.

Definition 2.5. The action α in Lemma 2.4(2),

$$\alpha_A(l_\Theta(x)) = l_\Theta(Ax), \quad A \in G, \quad x \in \mathbb{Z}^d,$$

is called the *canonical action* of $G(\subset G_\Theta)$ on \mathcal{A}_Θ .

Note that by (2.7) and Lemma 2.4(3), we have an isomorphism

$$\mathcal{A}_\Theta \rtimes_\alpha G \cong C^*(\mathbb{Z}^d \rtimes G, \tilde{\omega}_\Theta) \quad (2.9)$$

for any subgroup G of G_Θ and its canonical action α .

Example 2.6. For $\Theta = (\theta_{ij}) \in \mathcal{T}_2(\mathbb{R})$ with $\theta := \theta_{12}$, the canonical action α of $G_\Theta (= \text{SL}_2(\mathbb{Z}))$ by Remark 2.3(2)) coincides with the action in (1.2); if $A = (a_{ij}) \in \text{SL}_2(\mathbb{Z})$, we have for $i = 1, 2$,

$$\begin{aligned} \alpha_A(l_\Theta(e_i)) &= l_\Theta(Ae_i) = l_\Theta(a_{1i}e_1 + a_{2i}e_2) \\ &= \exp(\pi i \theta a_{1i} a_{2i}) l_\Theta(e_1)^{a_{1i}} l_\Theta(e_2)^{a_{2i}} \text{ (by (2.6))}. \end{aligned}$$

2.4. Companion matrices of cyclotomic polynomials. To find a finite subgroup G of $G_\Theta(\subset \text{GL}_d(\mathbb{Z}))$, we have to examine matrices $A \in \text{GL}_d(\mathbb{Z})$ of finite order. For this, we shall use companion matrices of cyclotomic polynomials in Section 4 and 5.

Consider a monic polynomial $p(x) = a_0 + a_1x + \cdots + a_{d-1}x^{d-1} + x^d$. The *companion matrix* $C_{p(x)}$ of $p(x)$ is defined to be the following $d \times d$ matrix

$$C_{p(x)} := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -a_{d-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{d-1} \end{pmatrix} \quad (2.10)$$

which is invertible if $a_0 \neq 0$. The minimal polynomial of $C_{p(x)}$ is equal to its characteristic polynomial $p(x)$.

Recall that for $n \in \mathbb{N}$, the n th *cyclotomic polynomial* $\Phi_n(x)$ is defined by

$$\Phi_n(x) = \prod_{\substack{1 \leq k \leq n \\ \gcd(k, n) = 1}} (x - \exp(2\pi i \frac{k}{n})).$$

It is a monic polynomial of degree $d := \phi(n)$ (here, ϕ is the Euler's totient function). $\Phi_n(x)$ is also known to have integer coefficients and is irreducible over

\mathbb{Q} . The companion matrix $C_n := C_{\Phi_n(x)}$ of $\Phi_n(x)$ is then a matrix of order n . If $n \geq 3$, then d is even and it is easy to see that $C_n \in \mathrm{SL}_d(\mathbb{Z})$, namely $\det(C_n) = 1$. Since the minimal polynomial $\Phi_n(x)$ of C_n has distinct roots $\{\exp(2\pi i \frac{k}{n}) : 1 \leq k \leq n, \gcd(k, n) = 1\}$ which are the eigenvalues of C_n , we see that C_n is diagonalizable (in \mathbb{C}). Thus there is an invertible matrix U such that

$$UC_n U^{-1} = \mathrm{diag}(\zeta_1, \dots, \zeta_{\phi(n)}), \quad (2.11)$$

where $\zeta_1, \dots, \zeta_{\phi(n)}$ are the distinct primitive n -th roots of unity.

Remark 2.7. Let $n \in \mathbb{N}$ and $d := \phi(n)$. Then the companion matrix C_n of the cyclotomic polynomial $\Phi_n(x)$ generates the finite group $\mathbb{Z}_n = \langle C_n \rangle := \{C_n^k \in \mathrm{GL}_d(\mathbb{Z}) : 0 \leq k \leq n-1\}$ which acts on the group \mathbb{Z}^d via

$$(C_n^k, x) \mapsto C_n^k x : \mathbb{Z}_n \times \mathbb{Z}^d \rightarrow \mathbb{Z}^d.$$

This action (also denoted α) is actually the conjugation action of \mathbb{Z}_n on \mathbb{Z}^d in the semidirect product group $\mathbb{Z}^d \rtimes_{\alpha} \mathbb{Z}_n$.

2.5. Classification theorems. Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ be nondegenerate. Then \mathcal{A}_{Θ} is a simple C^* -algebra with a unique tracial state by [8, Theorem 1.9] and has tracial rank zero by [8, Theorem 3.5]. Thus if $\alpha : G \rightarrow \mathrm{Aut}(\mathcal{A}_{\Theta})$ is an action by a finite group which has the tracial Rokhlin property (see [3, Section 5]), the crossed product $\mathcal{A}_{\Theta} \rtimes_{\alpha} G$ becomes a simple C^* -algebra ([9, Corollary 1.6]) with tracial rank zero ([9, Theorem 2.6]). The fact that $\mathcal{A}_{\Theta} \rtimes_{\alpha} G$ has a unique tracial state follows from [3, Proposition 5.7].

The canonical action α of a finite group $G(\subset G_{\Theta})$ on the simple \mathcal{A}_{Θ} is actually known to have the tracial Rokhlin property by [3, Lemma 5.10 and Theorem 5.5], and moreover the crossed product $\mathcal{A}_{\Theta} \rtimes_{\alpha} G$ satisfies the Universal Coefficient Theorem (this will be shown in Proposition 3.1). Thus the crossed product $\mathcal{A}_{\Theta} \rtimes_{\alpha} G$ becomes classifiable by Huaxin Lin's classification theorem:

Theorem 2.8. ([6, Theorem 5.2]) *Let A and B be two unital separable simple nuclear C^* -algebras with tracial topological rank zero which satisfy the Universal Coefficient Theorem. Then $A \cong B$ if and only if they have isomorphic Elliott invariants, that is,*

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)).$$

Since simple unital AF algebras satisfy all the conditions of the above theorem, if the Elliott invariant of $\mathcal{A}_{\Theta} \rtimes_{\alpha} G$, $G \subset G_{\Theta}$, is isomorphic to that of such an AF algebra, one can conclude that the crossed product $\mathcal{A}_{\Theta} \rtimes_{\alpha} G$ is an AF algebra, which was successfully done in [3] for $\mathcal{A}_{\theta} \rtimes_{\alpha} F$ with all $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and all finite subgroups F of $\mathrm{SL}_2(\mathbb{Z})$. The following proposition by N.C. Phillips also says that to see whether the crossed product $\mathcal{A}_{\Theta} \rtimes_{\alpha} G$ is AF, we only need to know its K -groups:

Proposition 2.9. ([8, Proposition 3.7]) *Let \mathcal{A} be a simple infinite dimensional separable unital nuclear C^* -algebra with tracial rank zero and which satisfies the*

Universal Coefficient Theorem. Then \mathcal{A} is a simple AH algebra with real rank zero and no dimension growth. If $K_*(\mathcal{A})$ is torsion free, \mathcal{A} is an AT algebra. If, in addition, $K_1(\mathcal{A}) = 0$, then \mathcal{A} is an AF algebra.

Remark 2.10. We can summarize what the classification results above together with Proposition 3.1 imply in our setting as follows: If Θ is a nondegenerate skew symmetric real $d \times d$ matrix and $\alpha : G \rightarrow \text{Aut}(\mathcal{A}_\Theta)$ is the canonical action of a finite group $G \subset G_\Theta$, the simple crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is an AF algebra if and only if $K_0(\mathcal{A}_\Theta \rtimes_\alpha G)$ is torsion free and $K_1(\mathcal{A}_\Theta \rtimes_\alpha G) = 0$.

For the computation of K_1 -groups of the crossed products $\mathcal{A}_\Theta \rtimes_\alpha G$, we shall apply the following theorem.

Theorem 2.11. ([5, Theorem 0.1], [2, Theorem 0.3]) *Let $n, d \in \mathbb{N}$. Consider the extension of groups $1 \rightarrow \mathbb{Z}^d \rightarrow \mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n \rightarrow \mathbb{Z}_n \rightarrow 1$ such that conjugation action α of \mathbb{Z}_n on \mathbb{Z}^d is free outside the origin $0 \in \mathbb{Z}^n$. Then $K_0(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) \cong \mathbb{Z}^{s_0}$ for some $s_0 \in \mathbb{Z}$ and*

$$K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) \cong \mathbb{Z}^{s_1},$$

where $s_1 = \sum_{l \geq 0} \text{rk}_{\mathbb{Z}}((\Lambda^{2l+1} \mathbb{Z}^d)^{\mathbb{Z}_n})$. If n is even, $s_1 = 0$. If $n > 2$ is prime and $d = n - 1$, then $s_1 = \frac{2^{n-1} - (n-1)^2}{2n}$.

3. K -GROUPS OF THE SIMPLE CROSSED PRODUCTS $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$

In this section we show that the K_1 -group of the simple crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is not always zero (see Theorem 3.6). We begin with the following proposition saying that we can apply Lin's classification theorem (Theorem 2.8) or Proposition 2.9 to the simple crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ of the canonical action by a finite subgroup G of G_Θ .

Proposition 3.1. *Let Θ be a skew symmetric real $d \times d$ matrix and G be a finite subgroup of G_Θ . If $\alpha : G \rightarrow \text{Aut}(\mathcal{A}_\Theta)$ is the canonical action of G on \mathcal{A}_Θ , the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ satisfies the Universal Coefficient Theorem.*

Proof. The 2-cocycle ω_Θ given in (2.4) is invariant under the action of G on \mathbb{Z}^d . By (2.9), the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is isomorphic to the twisted group algebra $C^*(\mathbb{Z}^d \rtimes G, \tilde{\omega}_\Theta)$. Note that $\mathbb{Z}^d \rtimes G$ is amenable and is a closed subgroup of $\mathbb{R}^d \rtimes G$ which is almost connected. Therefore $\mathcal{A}_\Theta \rtimes_\alpha G$ satisfies the Universal Coefficient Theorem (see [3, Corollary 6.2]). \square

Remark 3.2. If there is a finite subgroup G of G_Θ which canonically acts on the noncommutative simple torus \mathcal{A}_Θ , as summarized in Remark 2.10, we need to calculate the K -groups of $\mathcal{A}_\Theta \rtimes_\alpha G$ to see whether it is an AF algebra. But the K -groups $K_*(\mathcal{A}_\Theta \rtimes_\alpha G)$ of the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G \cong C^*(\mathbb{Z}^d \rtimes G, \tilde{\omega}_\Theta)$ (see (2.9)) is equal to the K -groups $K_*(C^*(\mathbb{Z}^d \rtimes G))$ of the untwisted group algebra

by [3, Theorem 0.3];

$$K_i(\mathcal{A}_\Theta \rtimes_\alpha G) = K_i(C^*(\mathbb{Z}^d \rtimes G))$$

for $i = 0, 1$. This is because the 2-cocycle $\tilde{\omega}_\Theta$ is homotopic (in the sense of [3, Theorem 0.3]) to the trivial one via

$$\Omega : (\mathbb{Z}^d \rtimes G) \times (\mathbb{Z}^d \rtimes G) \rightarrow C([0, 1], \mathbb{T})$$

defined by

$$\Omega((x, A), (y, B))(t) := \exp(2\pi i t \langle \Theta x, Ay \rangle)$$

for $x, y \in \mathbb{Z}^d$, $A, B \in G$ and $t \in [0, 1]$.

For the rest of this section, we will consider the cyclic group $\mathbb{Z}_n = \langle C_n \rangle$ generated by the companion matrix C_n of the n th cyclotomic polynomial and its conjugation action α on \mathbb{Z}^d , $d = \phi(n)$ (see Remark 2.7 for the conjugation action). Note that we use the same α for both the canonical action of $G(\subset G_\Theta)$ on \mathcal{A}_Θ and the conjugation action of \mathbb{Z}_n on \mathbb{Z}^d .

The formula $K_*(C^*(\mathbb{Z}^d \rtimes_\alpha G))$ in Theorem 2.11 requires that the action α of \mathbb{Z}_n on \mathbb{Z}^d be free outside the origin, so we need the following proposition which follows immediately from (2.11).

Proposition 3.3. *The conjugation action α of $\mathbb{Z}_n = \langle C_n \rangle$ on \mathbb{Z}^d in the semidirect product group $\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n$ is free outside the origin, that is, $C_n^k x \neq x$ for all $k = 1, \dots, n-1$ and nonzero $x \in \mathbb{Z}^d$.*

The above proposition, together with Proposition 2.9, Remark 3.2, and Theorem 2.11, gives the following:

Proposition 3.4. *Let $d = \phi(n)$ and α be the canonical action of the finite cyclic group $\mathbb{Z}_n = \langle C_n \rangle$ on a noncommutative simple d -torus \mathcal{A}_Θ . Then the crossed product $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is an AT algebra, and moreover it is an AF algebra if and only if $K_1(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n) = 0$; in particular if n is even, $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is always an AF algebra.*

Now let $n \geq 3$ be an odd number and α be the conjugation action of $\mathbb{Z}_n = \langle C_n \rangle$ on \mathbb{Z}^d . We will show that $K_1(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)$ is not necessarily zero. For each $l \geq 0$, α induces an action $\Lambda^l(\alpha) : \mathbb{Z}_n \rightarrow \text{Aut}(\Lambda^l \mathbb{Z}^d)$ of \mathbb{Z}_n on the l th exterior power $\Lambda^l \mathbb{Z}^d$ of \mathbb{Z} -module \mathbb{Z}^d as follows:

$$\begin{aligned} \Lambda^l(\alpha)(k)(x_1 \wedge \cdots \wedge x_l) &:= \Lambda^l(\alpha_k)(x_1 \wedge \cdots \wedge x_l) \\ &= \alpha_k(x_1) \wedge \cdots \wedge \alpha_k(x_l) \end{aligned}$$

for $k \in \mathbb{Z}_n$ and $x_1 \wedge \cdots \wedge x_l \in \Lambda^l \mathbb{Z}^d$. For notational convenience, we simply write $\Lambda(\alpha)$ for $\Lambda^l(\alpha)$. To compute $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n))$, we need to know the rank $\text{rk}_{\mathbb{Z}}(\Lambda^l \mathbb{Z}^d)^{\mathbb{Z}_n}$ of the following submodule

$$(\Lambda^l \mathbb{Z}^d)^{\mathbb{Z}_n} := \{v \in \Lambda^l \mathbb{Z}^d : \Lambda(\alpha_k)(v) = v, \ k \in \mathbb{Z}_n\}$$

of the fixed points.

Lemma 3.5. *Let $n \geq 3$ be an odd integer and $d := \phi(n)$ (automatically even). Then $\Lambda^d \mathbb{Z}^d = (\Lambda^d \mathbb{Z}^d)^{\mathbb{Z}_n}$.*

Proof. It is enough to show that $\Lambda(\alpha_k)(e_1 \wedge \cdots \wedge e_d) = e_1 \wedge \cdots \wedge e_d$ for all $k \in \mathbb{Z}_n$, which is obvious from $\Lambda(\alpha_k)(e_1 \wedge \cdots \wedge e_d) = \alpha_k(e_1) \wedge \cdots \wedge \alpha_k(e_d) = C_n^k e_1 \wedge \cdots \wedge C_n^k e_d = \det(C_n^k) e_1 \wedge \cdots \wedge e_d$ and the fact that $\det(C_n) = 1$. \square

For two subsets I, J of \mathbb{Z} , set

$$J - I := \{j - i \in \mathbb{Z} : i \in I, j \in J\}$$

and write $I \equiv J \pmod{n}$ if for each $i \in I$ there exists a $j \in J$ with $i - j \in n\mathbb{Z}$ and vice versa. As usual, $|I|$ denotes the cardinality of the set I .

Theorem 3.6. *Let $n \geq 7$ be an odd integer and $d := \phi(n)$. Consider the extension of groups $1 \rightarrow \mathbb{Z}^d \rightarrow \mathbb{Z}^d \rtimes_{\alpha} \mathbb{Z}_n \rightarrow \mathbb{Z}_n \rightarrow 1$ with $\mathbb{Z}_n = \langle C_n \rangle$. If $2d \geq n + 5$, then*

$$K_1(C^*(\mathbb{Z}^d \rtimes_{\alpha} \mathbb{Z}_n)) \neq 0.$$

(If $n \geq 7$ is prime, $2d \geq n + 5$ always holds.)

Proof. Since the conjugation action α of $\mathbb{Z}_n = \langle C_n \rangle$ on \mathbb{Z}^d is free outside the origin by Proposition 3.3, it is enough to show that $s_1 = \sum_{l \geq 0} \text{rk}_{\mathbb{Z}}((\Lambda^{2l+1} \mathbb{Z}^d)^{\mathbb{Z}_n}) \neq 0$ by Theorem 2.11.

Note first that the set $\{1, 2, \dots, d\}$ can be divided into two disjoint sets $I = \{i_1, \dots, i_l\}$ and $J = \{j_1, \dots, j_{d-l}\}$ such that $|I|$ (hence $|J|$) is odd and

$$\begin{aligned} J - I &\equiv \{1, 2, \dots, d-2\} \cup \{n-d+3, n-d+4, \dots, n-1\} \pmod{n} \\ &\equiv \{1, 2, \dots, n-1\} \pmod{n}, \end{aligned}$$

which follows from the condition $2d \geq n + 5$ (or $d - 2 \geq n - d + 3$). (For example, one can take $I = \{1, 2, d\}$ and $J = \{3, 4, \dots, d-1\}$.) Then for any $k, t \in \mathbb{Z}_n = \{0, 1, \dots, n-1\}$ with $k \neq t$, there exist $i \in I$ and $j \in J$ such that $k + i \equiv t + j \pmod{n}$. So we have

$$\begin{aligned} &\sum_{0 \leq k \neq t \leq n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \cdots \wedge e_{i_l}) \wedge \Lambda(\alpha_t)(e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}) \\ &= \sum_{0 \leq k \neq t \leq n-1} C_n^k e_{i_1} \wedge \cdots \wedge C_n^k e_{i_l} \wedge C_n^t e_{j_1} \wedge \cdots \wedge C_n^t e_{j_{d-l}} \\ &= \sum_{0 \leq k \neq t \leq n-1} C_n^{k+i_1-1} e_1 \wedge \cdots \wedge C_n^{k+i_l-1} e_1 \wedge C_n^{t+j_1-1} e_1 \wedge \cdots \wedge C_n^{t+j_{d-l}-1} e_1 \\ &= 0, \end{aligned}$$

where the second equality comes from the easy fact that $C_n^{v-1} e_1 = e_v$ for $v = 1, \dots, d$. By Lemma 3.5, $\Lambda(\alpha_k)(e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}) = e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge$

$e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}$ for all k . Thus

$$\begin{aligned}
0 &\neq n(e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}) \\
&= \sum_{k=0}^{n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}) \\
&= \left(\sum_{k=0}^{n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \cdots \wedge e_{i_l}) \right) \wedge \left(\sum_{t=0}^{n-1} \Lambda(\alpha_t)(e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}) \right) \\
&\quad - \sum_{0 \leq k \neq t \leq n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \cdots \wedge e_{i_l}) \wedge \Lambda(\alpha_t)(e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}) \\
&= \left(\sum_{k=0}^{n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \cdots \wedge e_{i_l}) \right) \wedge \left(\sum_{t=0}^{n-1} \Lambda(\alpha_t)(e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}) \right),
\end{aligned}$$

and hence $\sum_{k=0}^{n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \cdots \wedge e_{i_l}) \neq 0$ and $\sum_{t=0}^{n-1} \Lambda(\alpha_t)(e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}) \neq 0$. But clearly these two elements belong to $(\Lambda^l \mathbb{Z}^d)^{\mathbb{Z}_n}$ and $(\Lambda^{d-l} \mathbb{Z}^d)^{\mathbb{Z}_n}$ respectively, so that $s_1 > 0$ follows. \square

We close this section with providing a condition equivalent to $2d \geq n + 5$ used in Theorem 3.6.

Proposition 3.7. *Let $n \geq 7$ be an odd number and $d := \phi(n)$. Then the condition $2d \geq n + 5$ of Theorem 3.6 holds if and only if there is a partition $\{I, J\}$ of $\{1, \dots, d\}$ such that both $|I|$ and $|J|$ are odd and*

$$J - I \equiv \{1, 2, \dots, n-1\} \pmod{n}.$$

Proof. The direction ‘only if’ was proved in the proof of Theorem 3.6. For the converse, let $\{I, J\}$ be such a partition of $\{1, \dots, d\}$ that $|I|$ and $|J|$ are odd and $J - I \equiv \{1, 2, \dots, n-1\} \pmod{n}$. Assume $1 \in I$. Then the sets

$$P := (J - I) \cap \mathbb{N} \quad \text{and} \quad N := (J - I) \setminus P.$$

contain the positive and negative integers of $J - I$ respectively. With $N' := \{n + m : m \in N\}$ of positive integers, one has $N' \equiv N \pmod{n}$. Also

$$P \subset \{1, 2, \dots, d-1\} \quad \text{and} \quad N' \subset \{n + (1-d), n + (2-d), \dots, n-1\} \quad (3.12)$$

is clear, hence $P \cup N' \subset \{1, 2, \dots, n-1\}$ and $|P|, |N'| \leq d-1$. From

$$J - I = P \cup N \equiv P \cup N' \pmod{n} \quad \text{and} \quad J - I \equiv \{1, 2, \dots, n-1\} \pmod{n},$$

it follows that

$$P \cup N' = \{1, 2, \dots, n-1\} \quad (3.13)$$

since $P \cup N'$ has only positive integers less than n . Thus $n-1 \leq |P| + |N'| \leq 2(d-1)$, namely $2d \geq n+1$ must hold. Since n is odd, we have $2d = n+1$, $2d = n+3$ or $2d \geq n+5$.

To show $2d \geq n + 5$, first suppose $2d = n + 1$. Then $2(d - 1) = n - 1$ so that we have $N' = \{n + 1 - d, n + 2 - d, \dots, n - 1\}$ and thus $1 - d \in N$. But $1 - d$ is not equal to any number $j - i$ for $j \in J$ and $i \in I$. Thus $2d \neq n + 1$.

Now suppose that $2d = n + 3$, that is, $2(d - 1) = n + 1$ holds. Set

$$L := \{1, 2, \dots, d - 1\} \quad \text{and} \quad R := \{n + 1 - d, n + 2 - d, \dots, n - 1\},$$

then $2d = n + 3$ is the case exactly when

$$L \cap R = \{d - 2, d - 1\} = \{n + 1 - d, n + 2 - d\}. \quad (3.14)$$

Also (3.13) implies that

$$\{1, 2, \dots, d - 3\} \subset P \quad \text{and} \quad \{n + 3 - d, n + 4 - d, \dots, n - 1\} \subset N'. \quad (3.15)$$

Note here that if $m \in L \setminus R = \{1, \dots, d - 3\}$ then $m \in J - I$ and that if $m \in R \setminus L$ then $m - n \in J - I$. Moreover each $m \in L \cap R$ satisfies the following:

$$m \notin J - I \Rightarrow m - n \in J - I. \quad (3.16)$$

We claim that

$$\{1, d\} \subset I \quad \text{and} \quad \{2, d - 1\} \subset J.$$

First observe that for $m := n + 1 - d \in L \cap R$, $m - n = 1 - d \notin J - I$ since $1 \in I$. By (3.16), $m = n + 1 - d \in J - I$ and hence $d - 2 \in J - I$ by (3.14). Thus we have

$$(d, 2) \in J \times I \quad \text{or} \quad (d - 1, 1) \in J \times I. \quad (3.17)$$

Since $3 - d \in J - I$ by (3.15), we should have at least one of the following;

$$(3, d) \in J \times I, \quad (2, d - 1) \in J \times I, \quad \text{or} \quad (1, d - 2) \in J \times I.$$

But $(2, d - 1) \notin J \times I$ and $(1, d - 2) \notin J \times I$ because of (3.17) and our assumption that $1 \in I$. Thus $(3, d) \in J \times I$. Since $d \notin J$ we also have $d - 1 \in J$ by (3.17). From (3.14) $d - 1 = n + 2 - d$ and thus we have $n + 2 - d = d - 1 \notin J - I$ (otherwise, $d \notin J$). Then by (3.16), $2 - d \in J - I$ which can occur only when $(2, d) \in J \times I$ or $(1, d - 1) \in J \times I$. Since $1 \in I$, we obtain $2 \in J$. This proves the claim. Next we show that

$$\{1, d\} \subset I \quad \text{and} \quad \{2, 3, d - 2, d - 1\} \subset J$$

providing a proof that can be repeated until we reach the step $\{1, d\} = I$ and $\{2, \dots, d - 1\} = J$, where we meet a contradiction to the assumption that both I and J have odd number of elements. By (3.15), with $k = 2$, $n + (k + 1) - d \in N'$ and $d - (k + 1) \in P$. Thus

$$(k + 1) - d \in J - I \quad \text{and} \quad d - (k + 1) \in J - I.$$

Note that $(k + 1) - d = 3 - d \in J - I$ can happen only if $(k + 1, d) = (3, d) \in J \times I$, $(k, d - 1) = (2, d - 1) \in J \times I$, or $(1, d - k) = (1, d - 2) \in J \times I$. But obviously $(k + 1, d) = (3, d) \in J \times I$ is the only possible case and we have $k + 1 = 3 \in J$. Since $d - (k + 1) = d - 3 \in J - I$ we obtain $\{2, 3, d - 2, d - 1\} \in J$. (One can repeat the same argument on k .)

So far we have shown that the cases $2d = n + 1$ and $2d = n + 3$ should be excluded from $2d \geq n + 1$, which gives $2d \geq n + 5$ as desired. \square

4. FINITE CYCLIC GROUPS ACTING ON HIGHER DIMENSIONAL NONCOMMUTATIVE SIMPLE TORI \mathcal{A}_Θ

We have seen in the previous section that if α is a canonical action of $\mathbb{Z}_n = \langle C_n \rangle$ on a higher dimensional simple d -torus \mathcal{A}_Θ , then we are well informed about the crossed products $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ since we know how to compute the decisive invariants, namely the K -groups $K_*(\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n)$ which are equal to $K_*(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n))$ (Remark 3.2) and are not necessarily zero by Theorem 3.6. So it seems that many of the crossed products $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ are far from being AF. But what is not yet clear is if there does exist any noncommutative simple d -torus \mathcal{A}_Θ that actually admits the canonical action by a finite group. In this section we show that there do really exist such higher dimensional simple tori and then determine when their crossed products are not AF. To explain more precisely what we do here, we remark that we do not try to find finite groups G acting on a torus \mathcal{A}_Θ with Θ preassigned because this way of finding examples does not seem to work effectively, rather if we begin with a finite (cyclic) group G generated by a matrix $C \in \mathrm{GL}_d(\mathbb{Z})$ of finite order, it is more easily addressable to find d -tori \mathcal{A}_Θ on which G acts canonically. This is our strategy to investigate examples in this paper, and so we use the following notation for convenience' sake.

Notation 4.1. For $A \in \mathrm{GL}_d(\mathbb{Z})$, we set:

$$\mathcal{T}_{d,A}(\mathbb{R}) := \{\Theta \in \mathcal{T}_d(\mathbb{R}) : \Theta = A^t \Theta A\}.$$

$\mathcal{T}_{d,A}(\mathbb{R})$ is the set of all skew symmetric matrices Θ such that the noncommutative tori \mathcal{A}_Θ admit the canonical action of the group $\langle A \rangle$ generated by A . For $A \in \mathrm{GL}_d(\mathbb{Z})$ and $\Theta \in \mathcal{T}_d(\mathbb{R})$, it is obvious that $A \in G_\Theta$ if and only if $\Theta \in \mathcal{T}_{d,A}(\mathbb{R})$.

Since the companion matrix C_n is of order n , we see that any skew symmetric matrix $\Theta := \sum_{k=0}^{n-1} (C_n^k)^t \Theta' C_n^k$ (for $\Theta' \in \mathcal{T}_d(\mathbb{R})$) satisfies the condition $C_n^t \Theta C_n = \Theta$ to be an element of $\mathcal{T}_{d,C_n}(\mathbb{R})$. Conversely, if $\Theta \in \mathcal{T}_{d,C_n}(\mathbb{R})$, that is $C_n^t \Theta C_n = \Theta$, then $\Theta = \frac{1}{n} \sum_{k=0}^{n-1} (C_n^k)^t \Theta C_n^k$ is rather clear. Thus we see that

$$\mathcal{T}_{d,C_n}(\mathbb{R}) = \left\{ \sum_{k=0}^{n-1} (C_n^k)^t \Theta C_n^k : \Theta \in \mathcal{T}_d(\mathbb{R}) \right\}.$$

Now we show that for each $n \geq 3$, the cyclic group \mathbb{Z}_n acts canonically on some noncommutative simple $\phi(n)$ -dimensional tori \mathcal{A}_Θ . Note from the following theorem that since $\phi(n) = p_1^{r_1-1}(p_1-1) \cdots p_s^{r_s-1}(p_s-1)$ when $n = p_1^{r_1} \cdots p_s^{r_s}$ is the prime factorization of n , the group \mathbb{Z}_n always acts on some noncommutative higher dimensional simple tori whenever $n = 5$ or $n \geq 7$.

Theorem 4.2. *Let $n \geq 3$ and $d := \phi(n)$. Then there exist simple d -dimensional tori \mathcal{A}_Θ on which the group $\mathbb{Z}_n = \langle C_n \rangle$ acts canonically.*

Proof. We show that $\mathcal{T}_{d,C_n}(\mathbb{R})$ contains nondegenerate matrices. Let θ be an irrational number. Then $\Theta := \theta \sum_{k=0}^{n-1} (C_n^k)^t (C_n^t - C_n) C_n^k$ is a skew symmetric matrix in $\mathcal{T}_{d,C_n}(\mathbb{R})$. To show that Θ is nondegenerate, suppose $\Theta x \in \mathbb{Z}^d$ for some $x \in \mathbb{Z}^d$. Since the entries of C_n are integers and θ is irrational, we must have $(\sum_k (C_n^k)^t (C_n^t - C_n) C_n^k) x = 0$ in \mathbb{Z}^d . Then

$$\begin{aligned} 0 &= \left(\sum_{k=0}^{n-1} (C_n^k)^t (C_n^t - C_n) C_n^k \right) x \\ &= \left(C_n^t \sum_{k=0}^{n-1} (C_n^k)^t C_n^k - \left(\sum_{k=0}^{n-1} (C_n^k)^t C_n^k \right) C_n \right) x \\ &= \left(C_n^t \sum_{k=0}^{n-1} (C_n^k)^t C_n^k - C_n^t \left(\sum_{k=0}^{n-1} (C_n^k)^t C_n^k \right) C_n \right) x \\ &= C_n^t \sum_{k=0}^{n-1} (C_n^k)^t C_n^k (I_d - C_n^2) x, \end{aligned}$$

where I_d is the $d \times d$ identity matrix. Thus $(I_d - C_n^2)x$ must be zero because the matrix $C_n^t \sum_k (C_n^k)^t C_n^k$ is invertible. But, as we have seen in Proposition 3.3, $x \neq C_n^2 x$ for nonzero $x \in \mathbb{Z}^d$, and we conclude that Θ is nondegenerate. \square

In case that $n(\geq 3)$ is prime, we especially can find all the skew symmetric matrices $\Theta \in \mathcal{T}_{n-1}(\mathbb{R})$ such that the noncommutative tori \mathcal{A}_Θ associated with Θ admit the canonical action of the group $\mathbb{Z}_n = \langle C_n \rangle$. We begin with finding the general form of Θ for which the cyclic group generated by a companion matrix C of finite order can possibly act on \mathcal{A}_Θ .

Lemma 4.3. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ be a nonzero skew symmetric matrix and let $C \in \text{GL}_d(\mathbb{Z})$ be the companion matrix of a monic polynomial with degree $d(\geq 3)$. Assume that C is of order n and the noncommutative d -torus \mathcal{A}_Θ admits the canonical action by $\mathbb{Z}_n = \langle C \rangle$, then Θ has the following form:*

$$\Theta = \begin{pmatrix} 0 & \theta_0 & \theta_1 & \theta_2 & \cdots & \theta_{d-3} & \theta_{d-2} \\ & 0 & \theta_0 & \theta_1 & \theta_2 & \cdots & \theta_{d-3} \\ & & 0 & \theta_0 & \theta_1 & \ddots & \vdots \\ & & & 0 & \theta_0 & \ddots & \theta_2 \\ & & & & 0 & \ddots & \theta_1 \\ & & & & & \ddots & \theta_0 \\ & & & & & & 0 \end{pmatrix} \quad (4.18)$$

for $\theta_i \in \mathbb{R}$, $i = 0, \dots, d-2$.

Proof. Recall that $\mathbb{Z}_n = \langle C \rangle$ canonically acts on \mathcal{A}_Θ if and only if $C \in G_\Theta$, where $G_\Theta = \{A \in \text{GL}_d(\mathbb{Z}) : A^t \Theta A = \Theta\}$. Let Θ_i and Θ^j be the i -th row and j -th column of $\Theta = (\theta_{ij})$, respectively for $i, j = 1, \dots, d$. Let C be the companion matrix of a monic polynomial $a_1 + a_2x + \dots + a_dx^{d-1} + x^d$ of degree d ($a_1 \neq 0$ because C is invertible). Then we can write C (see (2.10)) as the sum $C = A + B$ of two matrices A and B , where

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & 0 & a_2 \\ 0 & 0 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_d \end{pmatrix}.$$

Then a computation, with $\mathbf{a} = (a_1, \dots, a_d)^t$, shows that

$$\begin{aligned} C^t \Theta C &= A^t \Theta A + A^t \Theta B + B^t \Theta A + B^t \Theta B \\ &= \begin{pmatrix} \theta_{22} & \cdots & \theta_{2d} & 0 \\ \vdots & \ddots & \vdots & 0 \\ \theta_{d2} & \cdots & \theta_{d,d} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & \Theta_2 \mathbf{a} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \Theta_d \mathbf{a} \\ 0 & \cdots & 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ (\Theta^2)^t \mathbf{a} & \cdots & (\Theta^d)^t \mathbf{a} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \mathbf{a}^t \Theta \mathbf{a} \end{pmatrix}. \end{aligned}$$

The assertion then follows from the fact that $C^t \Theta C$ is equal to Θ . \square

For a prime $p \geq 3$, consider the skew symmetric $(p-1) \times (p-1)$ matrices Θ of the following form:

$$\Theta = \begin{pmatrix} 0 & \theta_0 & \theta_1 & \theta_2 & \cdots & -\theta_2 & -\theta_1 \\ & 0 & \theta_0 & \theta_1 & \theta_2 & \cdots & -\theta_2 \\ & & 0 & \theta_0 & \theta_1 & \ddots & \vdots \\ & & & 0 & \theta_0 & \ddots & \theta_2 \\ & & & & 0 & \ddots & \theta_1 \\ & & & & & \ddots & \theta_0 \\ & & & & & & 0 \end{pmatrix}. \quad (4.19)$$

Theorem 4.4. *Let $p \geq 3$ be a prime number with $d = p - 1$. Then we have the following:*

- (1) *If a d -torus \mathcal{A}_Θ admits a canonical action of $\mathbb{Z}_p = \langle C_p \rangle$, namely $C_p^t \Theta C_p = \Theta$, then Θ must be of the form in (4.19).*
- (2) *If Θ is a skew symmetric matrix of form in (4.19), the group $\mathbb{Z}_p = \langle C_p \rangle$ canonically acts on \mathcal{A}_Θ .*

- (3) If Θ is a skew symmetric matrix of form in (4.19) such that the real numbers $1, \theta_0, \theta_1, \dots, \theta_{(p-3)/2}$ are independent over \mathbb{Z} , then the d -torus \mathcal{A}_Θ is simple.

Proof. For (1), first note that if Θ is a skew symmetric $d \times d$ matrix such that \mathcal{A}_Θ admits the canonical action of $\mathbb{Z}_p = \langle C_p \rangle$, then by Lemma 4.3, $\Theta = (\theta_{ij})$ must be of the form in (4.18). Since the p th cyclotomic polynomial is $\Phi_p(x) = 1 + x + \dots + x^{p-1}$, with $\mathbf{a} = (-1, \dots, -1)^t$, one can repeat the computation performed in the proof of Lemma 4.3 to obtain that

$$\begin{aligned} C_p^t \Theta C_p &= A^t \Theta A + A^t \Theta B + B^t \Theta A + B^t \Theta B \\ &= \begin{pmatrix} \theta_{22} & \cdots & \theta_{2d} & 0 \\ \vdots & \ddots & \vdots & 0 \\ \theta_{d2} & \cdots & \theta_{d,d} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & -\sum_j \theta_{2j} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -\sum_j \theta_{dj} \\ 0 & \cdots & 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ -\sum_i \theta_{i2} & \cdots & -\sum_i \theta_{id} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \sum_{i,j=1}^d \theta_{ij} \end{pmatrix}. \end{aligned}$$

But then from the fact that $C_p^t \Theta C_p$ is equal to the matrix

$$\Theta = \begin{pmatrix} 0 & \theta_{12} & \theta_{13} & \cdots & \theta_{1,p-2} & \theta_{1,p-1} \\ -\theta_{12} & 0 & \theta_{12} & \theta_{13} & \cdots & \theta_{1,p-2} \\ -\theta_{13} & -\theta_{12} & 0 & \theta_{12} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \theta_{13} \\ -\theta_{1,p-2} & & & & 0 & \theta_{12} \\ -\theta_{1,p-1} & -\theta_{1,p-2} & \cdots & -\theta_{13} & -\theta_{12} & 0 \end{pmatrix},$$

comparing the last columns of $C_p^t \Theta C_p$ and Θ , we have

$$\theta_{1,p-1} = -\sum_j \theta_{2j}, \quad \theta_{1,p-2} = -\sum_j \theta_{3j}, \quad \dots, \quad \theta_{13} = -\sum_j \theta_{d-1,j}, \quad \theta_{12} = -\sum_j \theta_{d,j}.$$

Note here that for any Θ of the above form, $\sum_j \theta_{kj} + \sum_j \theta_{d-(k-1),j} = 0$ for each $k = 1, \dots, d/2$, which shows

$$\theta_{1,p-1} = -\sum_j \theta_{2j} = \sum_j \theta_{d-1,j} = -\theta_{13}.$$

Similarly we see that $\theta_{1k} + \theta_{1,d-(k-1)} = 0$ for all $k = 3, \dots, d/2$.

It is just a simple observation from the above computation to see that $C_p^t \Theta C_p = \Theta$ holds for Θ in (4.19), thus (2) follows. Also, it is not hard to see that if $1, \theta_0, \theta_1, \dots, \theta_{(p-3)/2}$ are independent over \mathbb{Z} , then the skew symmetric matrix Θ is nondegenerate, which proves (3). \square

Corollary 4.5. *Let $p \geq 3$ be prime with $d = p - 1$ and Θ be a nondegenerate skew symmetric $d \times d$ matrix of the form in (4.19). Let $\alpha : \mathbb{Z}_p \rightarrow \text{Aut}(\mathcal{A}_\Theta)$ be the canonical action of $\mathbb{Z}_p = \langle C_p \rangle$ on the simple d -torus \mathcal{A}_Θ . Then the crossed product $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_p$ is an AF algebra if and only if $p = 3$ or 5 .*

Proof. By Proposition 3.4, we know that $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_p$ is AF if and only if $K_1(\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_p) = 0$, and this is the case when $p = 3$ or 5 from Theorem 2.11. \square

5. CANONICAL ACTIONS ON THREE DIMENSIONAL SIMPLE TORI

In the previous sections we considered the canonical action by the finite group \mathbb{Z}_n on noncommutative $\phi(n)$ -dimensional tori for $n \geq 3$. But $\phi(n)$ is always an even number for $n \geq 3$, thus the method using companion matrices won't work to find finite groups acting on odd-dimensional tori.

In this section we will focus on 3-dimensional tori and show that no simple noncommutative 3-tori admit the canonical actions of finite groups but the flip action: Recall that the *flip action* on a d -torus \mathcal{A}_Θ is the canonical action of $\mathbb{Z}_2 = \{I_d, -I_d\}$ generated by the flip automorphism which sends each generator $l_\Theta(e_i)$ to its adjoint $l_\Theta(e_i)^* = l_\Theta(-I_d e_i)$, $i = 1, \dots, d$.

Two group actions $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ and $\beta : H \rightarrow \text{Aut}(\mathcal{B})$ on C^* -algebras \mathcal{A} and \mathcal{B} are *conjugate* if there exist a group isomorphism $\psi : G \rightarrow H$ and a C^* -isomorphism $\rho : \mathcal{A} \rightarrow \mathcal{B}$ such that $\beta_{\psi(g)} \circ \rho = \rho \circ \alpha_g$ for all $g \in G$. This is an equivalence relation and two conjugate dynamical systems give rise to isomorphic crossed products. In the following proposition we provide a sufficient condition on two matrices $\Theta, \Theta' \in \mathcal{T}_d(\mathbb{R})$ that every canonical action on \mathcal{A}_Θ is conjugate to a canonical action on $\mathcal{A}_{\Theta'}$.

Proposition 5.1. *Let Θ and Θ' be two matrices in $\mathcal{T}_d(\mathbb{R})$ such that*

$$\Theta = (B^{-1})^t \Theta' B^{-1}$$

for some $B \in \text{GL}_d(\mathbb{Z})$. Then we have the following:

- (1) *The map $\psi : G_{\Theta'} \rightarrow G_\Theta$, $\psi(A) = BAB^{-1}$, is a group isomorphism.*
- (2) *If $\alpha : G' \rightarrow \text{Aut}(\mathcal{A}_{\Theta'})$ is a canonical action of a subgroup G' of $G_{\Theta'}$, it is conjugate to the canonical action $\beta : \psi(G') \rightarrow \text{Aut}(\mathcal{A}_\Theta)$ of $\psi(G')$.*

Proof. (1) Let $A \in G_{\Theta'}$. Then $\psi(A) = BAB^{-1} \in G_\Theta$ follows from

$$\begin{aligned} ((BAB^{-1})^{-1})^t \Theta (BAB^{-1})^{-1} &= (B^{-1})^t (A^{-1})^t B^t \Theta B A^{-1} B^{-1} \\ &= (B^{-1})^t (A^{-1})^t \Theta' A^{-1} B^{-1} \\ &= (B^{-1})^t \Theta' B^{-1} \\ &= \Theta. \end{aligned}$$

Clearly ψ is a group homomorphism with the inverse $A \mapsto B^{-1}AB$, $A \in G_\Theta$.

(2) One can check that $\rho := \text{Ad } U_B : \mathcal{A}_{\Theta'} \rightarrow \mathcal{A}_\Theta$ is an isomorphism such that $\rho(l_{\Theta'}(x)) = l_\Theta(Bx)$ for $x \in \mathbb{Z}^d$ (see Remark 2.3(1)), and then for $A \in G'$,

$$\rho \circ \alpha_A(l_{\Theta'}(x)) = \rho(l_{\Theta'}(Ax)) = l_\Theta(BAx).$$

On the other hand, since β is a canonical action, one also has

$$\beta_{\psi(A)} \circ \rho(l_{\Theta'}(x)) = \beta_{\psi(A)}(l_{\Theta}(Bx)) = l_{\Theta}(\psi(A)Bx) = l_{\Theta}(BAx)$$

for all $A \in G'$ and $x \in \mathbb{Z}^d$, which completes the proof. \square

As in the 2-dimensional case, if we have the complete list of group elements of finite order in $\mathrm{GL}_d(\mathbb{Z})$ up to conjugacy, then by Proposition 5.1 we would be able to find all canonical actions by finite cyclic groups on d -dimensional tori. Actually this is possible for $d = 3$ by virtue of the list (see Table 1) established in [14].

Theorem 5.2. *The only canonical action by a nontrivial finite cyclic group on a simple 3-dimensional torus is the flip action; if $A \in \mathrm{GL}_3(\mathbb{Z})$, $A \neq -I_3$, is a matrix in Table 1, every $\Theta \in \mathcal{T}_{3,A}(\mathbb{R})$ is degenerate.*

Proof. If a 3-torus $\mathcal{A}_{\Theta'}$ admits a canonical action of a finite cyclic group $G' \subset G_{\Theta'}$, then G' must be conjugate to a cyclic group generated by a matrix A in the table, so that there exists $B \in \mathrm{GL}_3(\mathbb{Z})$ such that $G' = \langle B^{-1}AB \rangle$ is generated by $B^{-1}AB$. But then by Proposition 5.1, the cyclic group $\langle A \rangle$ canonically acts on the 3-torus \mathcal{A}_{Θ} , where $\Theta := (B^{-1})^t \Theta' B^{-1}$. Also, it is rather obvious that Θ is nondegenerate exactly when Θ' is nondegenerate. Therefore by Theorem 2.2 it is enough to show that if A is one of the matrices listed in the table and $A \neq -I_3 (= A_5^2)$, every $\Theta \in \mathcal{T}_{3,A}(\mathbb{R})$ should be degenerate.

Recall that $\Theta \in \mathcal{T}_d(\mathbb{R})$ is degenerate if there exists a nonzero $x \in \mathbb{Z}^d$ such that $\exp(2\pi i \langle \Theta x, y \rangle) = 1$ for all $y \in \mathbb{Z}^d$, or equivalently if there is a nonzero $x \in \mathbb{Z}^d$ with $\langle \Theta x, e_j \rangle \in \mathbb{Z}$ for all $j = 1, \dots, d$. Thus, to obtain the degeneracy of $\Theta \in \mathcal{T}_{3,A}(\mathbb{R})$, we find nonzero elements $x \in \mathbb{Z}^3$ with $\Theta x \in \mathbb{Z}^3$.

It is rather tedious to do the same calculation with all the matrices in the table, so here we only do with $A = A_1^2$ and leave the rest to readers. If $\Theta =$

$$(\theta_{kj}) \in \mathcal{T}_{3,A}(\mathbb{R}), \text{ that is } \Theta - A^t \Theta A = \begin{pmatrix} 0 & 2\theta_{12} & 2\theta_{13} \\ -2\theta_{12} & 0 & 0 \\ -2\theta_{13} & 0 & 0 \end{pmatrix} \text{ is the zero matrix,}$$

then Θ must be of the form $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ 0 & -s & 0 \end{pmatrix}$ for an $s \in \mathbb{R}$. Any such matrix Θ is

degenerate; in fact, $\Theta x = (0, 0, 0)^t \in \mathbb{Z}^3$ for any $x = (k, 0, 0)^t \in \mathbb{Z}^3$. \square

6. CANONICAL ACTIONS ON FOUR DIMENSIONAL SIMPLE TORI

For the rest of the paper, we concretely examine examples of canonical actions on 4-dimensional tori \mathcal{A}_{Θ} by $\mathbb{Z}_n = \langle C_n \rangle$ with $\phi(n) = 4$.

Since $\phi(n) = p_1^{r_1-1}(p_1-1) \cdots p_s^{r_s-1}(p_s-1)$ when $n = p_1^{r_1} \cdots p_s^{r_s}$ is the prime factorization of n , $\phi(n) = 4$ implies that n should be equal to 5, 8, 10 or 12. For each of these n , since the n th cyclotomic polynomial is $\Phi_5(x) = 1+x+x^2+x^3+x^4$,

TABLE 1. Elements of finite order in $\mathrm{GL}_3(\mathbb{Z})$

order	generators
2	$A_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, A_2^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$ $A_3^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, A_4^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ & -1 & 0 \end{pmatrix},$ $A_5^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
3	$A_1^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, A_2^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
4	$A_1^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, A_2^4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$ $A_3^4 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, A_4^4 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$
6	$A_1^6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, A_2^6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix},$ $A_3^6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, A_4^6 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$

$\Phi_8(x) = 1+x^4$, $\Phi_{10}(x) = 1-x+x^2-x^3+x^4$, and $\Phi_{12}(x) = 1-x^2+x^4$, respectively,

the companion matrix C_n (of order n) of (2.10) is given by:

$$\begin{aligned} C_5 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad C_8 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ C_{10} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad C_{12} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (6.20)$$

Recall that the group $\mathbb{Z}_n = \langle C_n \rangle$ canonically acts on a 4-torus \mathcal{A}_Θ exactly when $C_n^t \Theta C_n = \Theta$, or equivalently when $\Theta \in \mathcal{T}_{4,C_n}(\mathbb{R})$. Since every skew symmetric matrix $\Theta \in \mathcal{T}_{4,C_n}(\mathbb{R})$ has the form in (4.18), by a simple computation we see that

$$\mathcal{T}_{4,C_n}(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \theta & \mu & \nu_n \\ & 0 & \theta & \mu \\ & & 0 & \theta \\ & & & 0 \end{pmatrix} : \theta, \mu \in \mathbb{R} \right\}, \quad (6.21)$$

where $\nu_5 = -\mu$, $\nu_8 = \theta$, $\nu_{10} = \mu$, and $\nu_{12} = 2\theta$.

Moreover, $\Theta \in \mathcal{T}_{4,C_n}(\mathbb{R})$ is easily seen to be nondegenerate whenever 1, θ , μ are independent over \mathbb{Z} .

Proposition 6.1. *If Θ is a skew symmetric 4×4 matrix in $\mathcal{T}_{4,C_n}(\mathbb{R})$ for some $n = 5, 8, 10, 12$, then the noncommutative 4-torus \mathcal{A}_Θ admits the canonical action α by the finite group $\mathbb{Z}_n = \langle C_n \rangle$ generated by C_n in (6.20). If $\Theta \in \mathcal{T}_{4,C_n}(\mathbb{R})$ is nondegenerate, the crossed product $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is an AF algebra.*

Proof. For n even, $K_1(\mathbb{Z}^4 \rtimes_\alpha \mathbb{Z}_n) = 0$ by Theorem 2.11 and $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is AF by Proposition 3.4. $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_5$ is AF by Corollary 4.5. \square

Remark 6.2. Let $m = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ be the prime factorization of an integer $m \in \mathbb{N}$, where $p_1 < p_2 < \cdots < p_t$ are primes. Then it is known ([4, Theorem 2.7]) that the group $\text{GL}_n(\mathbb{Z})$ has an element of order m if and only if

- (1) $\sum_{i=1}^t (p_i - 1) p_i^{k_i - 1} - 1 \leq n$ for $p_1^{k_1} = 2$, or
- (2) $\sum_{i=1}^t (p_i - 1) p_i^{k_i - 1} \leq n$ otherwise.

Thus, with $n = 4$, we see that any possible finite order of a matrix in $\text{GL}_4(\mathbb{Z}) \setminus \{I_4\}$ is one of 2, 3, 4, 5, 6, 8, 10, 12. It should be noted that the action by \mathbb{Z}_5 is not conjugate to any product action (on $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$) of two canonical actions by finite cyclic subgroups $F (\subset \text{SL}_2(\mathbb{Z}))$ on \mathcal{A}_θ because F is necessarily isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, or \mathbb{Z}_6 . The actions by these F are the only finite group actions on noncommutative tori found in the literature at least to the knowledge of the authors, which led us to work on finite group actions on higher dimensional tori.

Now we consider 4-dimensional noncommutative tori that are isomorphic to the tensor product $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ of an irrational rotation algebra \mathcal{A}_θ with itself. If

\mathcal{A}_Θ is associated with the following skew symmetric matrix

$$\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}, \quad (6.22)$$

it is easily seen to be isomorphic to the tensor product $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$. Moreover the following skew symmetric matrices $\Theta_{n,\theta}$,

$$\Theta_{5,\theta} = \Theta_{10,\theta} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & \theta & 0 \\ 0 & -\theta & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix} \in \mathcal{T}_{4,C_5}(\mathbb{R}) \cap \mathcal{T}_{4,C_{10}}(\mathbb{R}), \quad (6.23)$$

$$\Theta_{8,\theta} = \Theta_{12,\theta} = \begin{pmatrix} 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & \theta \\ -\theta & 0 & 0 & 0 \\ 0 & -\theta & 0 & 0 \end{pmatrix} \in \mathcal{T}_{4,C_8}(\mathbb{R}) \cap \mathcal{T}_{4,C_{12}}(\mathbb{R}), \quad (6.24)$$

(see (6.21)) give rise to 4-dimensional tori isomorphic to $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$:

Lemma 6.3. *Let Θ be the matrix in (6.22) with $\theta \in \mathbb{R}$ and $\Theta_{n,\theta}$ be one of the matrices in (6.23) or (6.24) for $n = 5, 8, 10, 12$. We then have the following:*

- (1) *There exists $B_n \in \text{GL}_n(\mathbb{Z})$ with $B_n^t \Theta_{n,\theta} B_n = \Theta$.*
- (2) *$G_\Theta = B_n^{-1} G_{\Theta_{n,\theta}} B_n$.*
- (3) *$\mathcal{A}_{\Theta_{n,\theta}}$ is isomorphic to \mathcal{A}_Θ .*

Proof. (1) The following B_n is the desired matrix for each n :

$$B_5 = B_{10} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad B_8 = B_{12} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(2) and (3) then follow from Proposition 5.1 and its proof. \square

Since, in the above situation, $C_n \in G_{\Theta_{n,\theta}}$, for $\theta \in \mathbb{R}$ and $n = 5, 8, 10, 12$, and

$$C \mapsto (B_n)^{-1} C B_n : G_{\Theta_{n,\theta}} \rightarrow G_\Theta$$

is a group isomorphism, we see that

$$A_n := (B_n)^{-1} C_n B_n$$

are the matrices (acting on \mathcal{A}_Θ) of order n for $n = 5, 8, 10, 12$ by Lemma 6.3(ii), and actually given by

$$\begin{aligned} A_5 &= \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ A_{10} &= \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (6.25)$$

Proposition 6.4. *Every 4-dimensional noncommutative torus of the form $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ admits a canonical action α_n of $\mathbb{Z}_n = \langle A_n \rangle$ in (6.25) for $n = 5, 8, 10, 12$. More precisely, if $\mathcal{A}_\theta \otimes \mathcal{A}_\theta = C^*(u_1, u_2) \otimes C^*(u_3, u_4)$ is generated by 4 unitaries u_i 's satisfying $u_2 u_1 = \exp(2\pi i \theta) u_1 u_2$, $u_4 u_3 = \exp(2\pi i \theta) u_3 u_4$, and $u_k u_l = u_l u_k$ for $k = 1, 2$ and $l = 3, 4$, we have the following:*

(1) $\alpha_5 : \mathbb{Z}_5 \rightarrow \text{Aut}(\mathcal{A}_\theta \otimes \mathcal{A}_\theta)$ is induced by the automorphism;

$$\begin{aligned} u_1 &\mapsto u_2 u_4^*, \quad u_2 \mapsto \exp(\pi i \theta) u_1^* u_2^*, \\ u_3 &\mapsto u_4, \quad u_4 \mapsto \exp(\pi i \theta) u_1^* u_2^* u_3^*. \end{aligned} \quad (6.26)$$

(2) $\alpha_8 : \mathbb{Z}_8 \rightarrow \text{Aut}(\mathcal{A}_\theta \otimes \mathcal{A}_\theta)$ is induced by the automorphism;

$$\begin{aligned} u_1 &\mapsto u_3, \quad u_2 \mapsto u_4, \\ u_3 &\mapsto u_2, \quad u_4 \mapsto u_1^*. \end{aligned} \quad (6.27)$$

(3) $\alpha_{10} : \mathbb{Z}_{10} \rightarrow \text{Aut}(\mathcal{A}_\theta \otimes \mathcal{A}_\theta)$ is induced by the automorphism;

$$\begin{aligned} u_1 &\mapsto u_2 u_4^*, \quad u_2 \mapsto \exp(-\pi i \theta) u_1^* u_2, \\ u_3 &\mapsto u_4, \quad u_4 \mapsto \exp(-\pi i \theta) u_1^* u_2 u_3^*. \end{aligned} \quad (6.28)$$

(4) $\alpha_{12} : \mathbb{Z}_{12} \rightarrow \text{Aut}(\mathcal{A}_\theta \otimes \mathcal{A}_\theta)$ is induced by the automorphism;

$$\begin{aligned} u_1 &\mapsto u_3, \quad u_2 \mapsto u_4, \\ u_3 &\mapsto u_2, \quad u_4 \mapsto \exp(-\pi i \theta) u_1^* u_2. \end{aligned} \quad (6.29)$$

Proof. We only show the case (1) here because the rest can be done similarly. Since \mathcal{A}_Θ is the twisted group algebra $C^*(\mathbb{Z}^4, \omega_\Theta) = C^*\{l_\Theta(e_i) : 1 \leq i \leq 4\}$ (2.7) with the identification $u_i := l_\Theta(e_i)$ for each i , the action α_5 of $\mathbb{Z}_5 = \langle A_5 \rangle$ is determined by $\alpha_5(k)(l_\Theta(e_i))$ for $k \in \mathbb{Z}_5$ and $1 \leq i \leq 4$. For convenience, we write

simply α for $\alpha_5(1)$ to have:

$$\begin{aligned}
\alpha(l_\Theta(e_1)) &= l_\Theta(A_5 e_1) = l_\Theta(e_2 - e_4) \\
&= \overline{\omega_\Theta(e_2, -e_4)} l_\Theta(e_2) l_\Theta(e_4)^* \\
&= \exp(-\pi i \langle \Theta e_2, -e_4 \rangle) l_\Theta(e_2) l_\Theta(e_4)^* \\
&= l_\Theta(e_2) l_\Theta(e_4)^*, \\
\alpha(l_\Theta(e_2)) &= l_\Theta(A_5 e_2) = \exp(\pi i \theta) l_\Theta(e_1)^* l_\Theta(e_2)^*, \\
\alpha(l_\Theta(e_3)) &= l_\Theta(A_5 e_3) = l_\Theta(e_4), \\
\alpha(l_\Theta(e_4)) &= l_\Theta(A_5 e_4) = \exp(\pi i \theta) l_\Theta(e_1)^* l_\Theta(e_2)^* l_\Theta(e_3)^*.
\end{aligned}$$

Thus α_5 is the action of \mathbb{Z}_5 generated by the automorphism α sending u_1 to $u_2 u_4^*$ and so on as stated in (6.26). \square

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